Indian Statistical Institute, Bangalore
M. Math. First Year
First Semester - Analysis of Several Variables

Mid-Semester Exam Date : Sept 10, 2015

Answer any five and each question carries 8 marks. Total marks: 40

E is an open subset of $\mathbb{R}^n$ unless otherwise mentioned.

1. (a) Let $f\colon E \to \mathbb{R}^m$ be differentiable at $x \in E$. Prove that f is continuous at x.
   (b) Prove that composite of two differentiable function is differentiable.

2. (a) Let $f\colon E \to \mathbb{R}$ be such that $D_1 f$, $D_2 f$ exist on E and $D_{21} f$ exists and continuous at some $v \in E$. Prove that $D_{12} f$ exists at v and $D_{12} f(v) = D_{21} f(v)$.
   (b) If $f\colon E \to \mathbb{R}$ has bounded partial derivatives, prove that f is continuous.

3. (a) Let E be a convex open set in $\mathbb{R}^n$ and $a, b \in E$. If $f\colon E \to \mathbb{R}$ is a differentiable function and $f'(u)(a) = f'(u)(b)$ for all $u \in E$, prove that $f(a) = f(b)$.
   (b) Let $f\colon E \to \mathbb{R}^m$ be a differentiable map with $f'(x) = 0$. Prove that $f^{-1}(\{y\})$ is open for any $y \in \mathbb{R}^m$ (*Marks: 3*).

4. Let $f\colon E \to \mathbb{R}^m$ have co-ordinate functions $f_1, f_2, \cdots, f_m$. Suppose $D_j f_i$ exists and continuous on E. Prove that f is differentiable. For $n > 1$, is the converse true? Justify your answer.

5. (a) If $f\colon \mathbb{R}^2 \to \mathbb{R}$ has continuous bounded second order partial derivatives, does $\lim_{||x|| \to \infty} \frac{f(x)}{||x||^3}$ exist? Justify your answer.
   (b) Let $f\colon E \to \mathbb{R}^n$ have continuous partial derivatives and $J_f(x_0) \neq 0$ for some $x_0 \in E$. Prove that f is one-one on an open subset N of E containing x_0.

6. (a) Find local maxi/mini of $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2$.
   (b) If $f\colon E \to \mathbb{R}$ is differentiable and $f(tx) = tf(x)$ for all $t \in \mathbb{R}$ and $x \in E$ with $tx \in E$, prove that $f(x) = \Sigma x_i D_i f(x)$ for all $x = (x_1, ..., x_n) \in E$.

7. (a) Can $x = u^2 - v^2$, $y = 2uv$ be solved for u and v in a neighborhood of $(u, v) = (1, 1)$. Justify your answer (*Marks: 3*).
   (b) Can the system of equations: $3x + y - z + u^2 = 0$; $x - y + 2z + u = 0$ and $2x + 2y - 3z + 2u = 0$ be solved for x, y, u in terms z.